Math 3435: Things to know for the midterm

1. How to determine if a PDE is nonlinear, linear homogeneous, linear inhomogeneous.
2. How linearity of a PDE affects solutions (that you can add solutions to get new solutions, find solution to an inhomogeneous linear PDE as a sum of a particular solution and solutions to the corresponding homogeneous PDE, etc.).
3. How to solve a first order linear PDE by using characteristics. The book (Section 1.2) gives what it calls the “geometric method” and the “coordinate method” – knowing either one is fine. You should be able to do constant and variable coefficient cases, and be able to find a solution given initial data.
4. What well-posed means (existence, uniqueness and stability of solutions). You should also know that equations can be well-posed for a region and ill-posed for another region (eg heat equation is well-posed for $t > 0$ and ill-posed for $t < 0$). You should know that there are different ways to measure stability, but you do not need to know the two ways we measured distance between data and solutions – if you were asked to say anything about stability you would be given information about the distance in the problem.
5. The basic second order equations: heat/diffusion $u_t = ku_{xx}$, wave $u_{tt} = c^2 u_{xx}$, and Laplace $u_{xx} = 0$. The fact that time-invariant solutions of heat and wave equations are solutions of the Laplace equation. The fact that derivatives of solutions to these equations are also solutions.
6. The fact that all linear 2nd order PDE are essentially equivalent to one of the three basic equations, which are referred to as parabolic (heat eqn), hyperbolic (wave) and elliptic (Laplace). You do not need to be able to do the change of variables that shows this.
7. For a given 2nd order linear PDE, how to determine whether it is parabolic, hyperbolic or elliptic using the sign of the determinant of the matrix of 2nd order coefficients (or an equivalent formula). You should be able to do this on the whole plane in the constant coefficient case and also be able to find regions of the plane on which different behaviors occur in the variable coefficient case.
8. The general solution formula $u(x,t) = f(x + ct) + g(x - ct)$ for the wave equation, and that (if $c > 0$) the term $f(x+ct)$ represents a wave moving left and $g(x-ct)$ a wave moving right. You should be able to use this equation to solve problems, but you do not need to know how to derive from it the formula for the solution with given initial data.
9. How to use the formula for the solution of the wave equation with given initial data. The formula will be on the formula sheet (last page of this document).
10. The fact that information propagated by the wave equation moves at speed $\leq c$. You need to know how to use this in problems and how it is expressed in the formula for the solution from initial data.
11. How to draw graphs of solutions of the wave equation from initial data.
12. That the wave equation conserves energy in the sense that $E'(t) = 0$ (see formula sheet for the energy $E(t)$), and how to verify this using the equation and integration by parts.
13. How to use energy to show that solutions of the wave equation are unique.
14. What the maximum principle for the heat equation says, and how to use it to show that a solution is bounded from bounds on its initial and boundary data. You do not need to know the proof of the maximum principle.
15. How to use the maximum principle to show that solutions of the heat equation are unique.
16. How to use energy in the heat equation to show that solutions are unique.
17. How to use the solution formula for the heat equation to solve problems when given initial data. How to write solutions in terms of the error function.

18. The following basic features of the heat and wave equations. Wave equation has finite propagation speed, information is transported, it is well-posed for all time, has energy conservation, and singularities move along characteristic lines. Heat equation has infinite propagation speed, information is lost, it is well-posed for $t > 0$ and ill-posed for $t < 0$, energy and solutions decay in time and are instantly smooth (there are no singularities).

19. How to use odd or even reflection to solve the heat equation on $(0, \infty)$ with initial data and Dirichlet or Neumann boundary data. How to write these solutions in terms of the error function.

20. How to use odd or even reflection to solve the wave equation on $(0, \infty)$ with initial data and Dirichlet or Neumann boundary data. How to graph these solutions.

21. How to solve inhomogeneous heat and wave equations on $\mathbb{R}$ and the half-line $(0, \infty)$. How to solve heat and wave equations with inhomogeneous boundary data by subtracting suitable functions to reduce to the case of an inhomogeneous PDE with homogeneous (Dirichlet or Neumann) boundary data.

22. How to perform separation of variables to convert a PDE to an eigenvalue problem.

23. How to solve the eigenvalue problem $X_n'' = -\lambda_n X_n$ for various boundary conditions. This was built up over several sections in Chapters 4 and 5 in the book, but at the end you should know that:

    (a) If the boundary conditions are symmetric, meaning that for f and g satisfying the boundary conditions we have $f'(x)g(x) - f(x)g'(x)\Big|_0^l = 0$, then the eigenvalues are real, the eigenfunctions can be chosen to be real, and the eigenfunctions for distinct eigenvalues are orthogonal. The computation that justifies this is $(\lambda_1 - \lambda_2)\int_0^l fg = \int_0^l fg'' - f''g = fg' - fg'\Big|_0^l = 0$. This happens in particular for Dirichlet, Neumann, Periodic, and Robin boundary conditions.

    (b) When the eigenfunctions X_n are orthogonal one can find the coefficients of $h(x) = \sum_0^\infty a_n X_n$ from $a_n = \langle h, X_n\rangle / \langle X_n, X_n\rangle$. It is useful to know the values of $\langle X_n, X_n\rangle$ for the standard (Dirichlet, Neumann, Periodic, Robin) cases.

    (c) If the boundary conditions are Dirichlet or Neuman or mixed Dirichlet and Neumann (i.e. Dirichlet at one end, Neumann at the other) then the eigenvalues can only be positive and the eigenfunctions are trigonometric with frequency $\sqrt{\lambda_n}$. The same is true for periodic boundary conditions. You should know what these eigenfunctions and eigenvalues are or how to find them explicitly. In the periodic case you should know the complex form as well as the real form.

    (d) If the boundary conditions are Robin there can be 0,1 or 2 non-positive eigenvalues, but you only need to know the conditions $a_0 + a_l = -a_0 a_l$ for a zero eigenvalue, $0 < a_0 + a_l < -a_0 a_l$ for exactly one negative eigenvalue and $a_0 + a_l > -a_0 a_l$ for no negative eigenvalues. All the rest are positive, and in fact λ_n is in $(n^2\pi^2 l^{-2}, (n+1)^2\pi^2 l^{-2})$ with $\lambda_n - n^2\pi^2 l^{-2} \to 0$ as $n \to \infty$. Equations for the eigenfunctions and eigenvalues are on the formula sheet.

24. A sine expansion of f on $[0, l]$ is equivalent to taking the odd reflection to $[-l, l]$ then the $2l$-periodic extension to $\mathbb{R}$, then expanding this function on $\mathbb{R}$. The cosine expansion is the same but for the even reflection of f to $[-l, l]$, and the "full Fourier" expansion is simply the $2l$-periodic extension of f from $[-l, l]$. These are continuous if f vanishes at the endpoints (sine series, Dirichlet case), f' vanishes at the endpoints (cosine series, Neumann case), or f is periodic (full Fourier series, periodic case), respectively.

25. We determined some conditions under which the Fourier series $\sum_{n=0}^\infty \frac{\langle f, X_n\rangle}{\langle X_n, X_n\rangle} X_n$ for a function f converges, for three different notions of convergence. Let f_N be the N^{th} partial sum of this series. Then the conditions are:

    (a) If f, f' and f'' exist and are continuous on the interval and f satisfies the boundary conditions for the eigenfunctions then the Fourier series converges to f uniformly, meaning that the maximum of $|f - f_N|$ over the interval goes to 0 as $N \to \infty$.

    (b) If $\int |f|^2 < \infty$ on the interval then $f_N \to f$ in L^2, meaning that $\int |f - f_N|^2 \to 0$ as $N \to \infty$. This does not imply convergence at any particular point.

(c) If f and f' are piecewise continuous on the interval and the series is one of the sine, cosine or full Fourier series then f_N converges pointwise. The limit of $f_N(x)$ is $\frac{1}{2}(\lim_{y\to x-} f(y) + \lim_{y\to x+} f(y))$. In particular, at continuity points of f, $f_N(x) \to f(x)$. Note that at the boundary points the limit may be determined using the odd, even, or periodic extension of f, so if f does not satisfy the boundary conditions of the series then the series will not converge at the boundary.

(d) At a jump discontinuity point, which may arise at a boundary point if f does not satisfy the boundary conditions of the eigenfunctions X_n, we see a Gibbs phenomenon. In this case the values of $f_N(x)$ differ from those of f by a fixed multiple of the jump at some point near the jump (so uniform convergence fails), and as $N \to \infty$ the point at which the Gibbs phenomenon occurs moves toward the jump location (permitting pointwise convergence to hold).

(e) A basic tool for analyzing the behavior of the sum of the Fourier series is the Dirichlet kernel $K_N(\theta) = \sin((N + \frac{1}{2})\theta)/\sin(\theta/2)$, because the values of $f_N(x)$ can be written using an integral of f against a translate and rescaling of K_N. In the case of the full Fourier series on $(-\pi, \pi)$ we have $f_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x-y)f(y)dy$. The function $K_N(\theta)$ has $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta)d\theta = 1$.

26. Since we can Fourier expand a piecewise differentiable function with respect to our basis of eigenfunctions, we can always expand $u(x,t) = \sum_n a_n(t)X_n(x)$ for each t for a suitable basis X_n. As far as solving the PDE goes, there are then two cases

   (a) If $u(x,t)$ satisfies the same (homogeneous) boundary conditions as $X_n(x)$ for each t we can simply differentiate inside the series to get an ODE for $a_n(t)$. This was, in effect, what happened in Chapter 4, where we simply wrote the series as $\sum_n c_n T_n(t)X_n(x)$, where $T_n(t)$ was the solution of the ODE in t that came from separation of variables, and the c_n came from the Fourier expansion of the initial data as in the first few sections of Chapter 5.

   (b) If the boundary conditions for u are not the same as those for X_n then the previous method fails, but we can still expand each of the functions and derivatives in the PDE and integrate by parts in the Fourier coefficient formulas for the x derivatives to get (generally inhomogeneous) ODEs for the $a_n(t)$. Solving these gives a series solution for the PDE. In general, our approach to these problems was to:

      i. Given a PDE with inhomogeneous symmetric boundary conditions, take at each boundary point the corresponding homogeneous condition and compute the corresponding basis X_n of eigenfunctions.

      ii. Expand all derivatives with respect to this basis X_n and write the PDE in terms of the coefficients in these expansions.

      iii. Compute the coefficients for the expansions using the formula for Fourier coefficients, integrating by parts and using the given inhomogeneous boundary conditions to obtain the boundary terms in these computations.

      iv. Combine the last two steps to give ODEs for the coefficients in the expansion of the solution function and solve these ODEs. Usually the initial data for the ODEs will be obtained by Fourier expansion of the initial data for the PDE.

      v. Substitute the ODE solutions into the series to obtain the solution to the PDE.

27. It is also possible sometimes to modify an inhomogeneous PDE or a PDE with inhomogeneous boundary data by subtracting a simple function or a known solution or similar so as to make the inhomogeneity or boundary data simpler. This is not critical to know, but it could save you a lot of time on some problems.

28. The Laplace equation $\Delta u = 0$ has a maximum principle. This says that if $\Delta u = 0$ on a bounded connected set D and is continuous on $D \cup \partial D$ then both the maximum and minimum of u on D are achieved on ∂D. Moreover, if either the maximum or the minimum of u are achieved inside D then u must be constant. Note that this result applies to solution of the Laplace equation, not the Poisson equation.

29. As a consequence of the maximum principle for the Laplace equation, on a bounded connected open set D there is at most one solution to $\Delta u = f$ on D, $u = h$ on ∂D where u is continuous on $D \cup \partial D$. The maximum principle also implies a stability result saying that if $\Delta u_1 = f = \Delta u_2$ and $u_1 - u_2$ is small on the boundary then it is small everywhere, because the maximum of $|u_1 - u_2|$ occurs on the boundary.

30. How to solve the Laplace and Poisson equations on a rotationally symmetric region (disc or annulus in $\mathbb{R}^2$ and ball or spherical annulus in $\mathbb{R}^3$) by using polar coordinates to reduce to an ODE. The polar form of Δ is on the formula sheet. Note that when doing a disc or ball there is implicitly a boundedness/continuity boundary condition at $r = 0$.

31. How to solve Laplace and Poisson equations on rectangular regions in two and three dimensions. The method is similar to that in 26(a), with an extra initial step.

    (a) Break u into a sum ($u_1 + u_2$ in dimension 2 or $u_1 + u_2 + u_3$ in dimension 3) so that for u_j the boundary conditions are homogeneous in all but the j^{th} direction.

    (b) Solve for each u_j by expanding in Fourier series for each of the directions with homogeneous boundary conditions (in the $\mathbb{R}^3$ case this is a double Fourier series); This involves some of the steps from 23 above.

    (c) For each u_j, solve for the direction with inhomogeneous boundary conditions to get a series for u_j; you then have a series for each u_j, and adding them gives a solution for the Laplace equation.

    (d) For example, for u_1 in $\mathbb{R}^2$ you have $u_1 = \sum_n X_n(x)Y_n(y)$ where $Y_n(y)$ are the eigenfunctions with eigenvalues λ_n. Solve for $X_n(x)$ using $X'' = \lambda_n X$ from the separated variables (obtaining hyperbolic functions); you will need to use the inhomogeneous boundary data on the $x = 0$, $x = a$ boundaries and Fourier wrt the y variable to get the endpoint data for this ODE.

    (e) Similarly, for u_1 in $\mathbb{R}^3$ you have $u_1 = \sum_n \sum_m X_{n,m}(x)Y_n(y)Z_m(z)$ with $Y_n(y)$, λ_n the eigenfunctions and eigenvalues for the y direction and $Z_m(z)$, μ_m those for the z-direction. This is a double Fourier series and you must solve $X''_{n,m} = (\lambda_n + \mu_m)X_{n,m}$ with endpoint data from decomposing the boundary values on $x = 0$ and $x = a$ into double Fourier series in y and z.

Math 3435: Formula sheet

Homogeneous wave equation on $\mathbb{R}$.

$$u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy$$

Homogeneous wave equation on $(0,\infty)$ with Dirichlet boundary data if $0 < x < ct$.

$$u(x,t) = \frac{1}{2}(\phi(x+ct) - \phi(ct-x)) + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)dy$$

Wave equation on $(0,\infty)$ with Neumann boundary data if $0 < x < ct$

$$u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(ct-x)) + \frac{1}{c}\int_{0}^{ct-x} \psi(y)dy + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)dy$$

Energy in wave equation

$$E(t) = \int u_t^2(y,t) + c^2 u_x^2(y,t)\,dy$$

Inhomogeneity in wave equation

Add $\frac{1}{2c}\iint_D f$, where D is the domain of dependence, for inhomogeneity in the PDE

Add $h(t - x/t)$, for a suitable h to remove an inhomogeneity in the boundary data.

Fundamental solution (or source) for heat equation if $t > 0$

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}$$

Inhomogeneous heat equation on $\mathbb{R}$ for $t > 0$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s)\,dy\,ds$$

Homogeneous heat equation on $(0,\infty)$ with Dirichlet boundary data for $t > 0$

$$u(x,t) = \int_0^{\infty} \Big(S(x-y,t) - S(x+y,t)\Big)\phi(y)\,dy$$

Homogeneous heat equation on $(0,\infty)$ with Neumann boundary data for $t > 0$

$$u(x,t) = \int_0^{\infty} \Big(S(x-y,t) - S(x+y,t)\Big)\phi(y)\,dy$$

Energy in heat equation

$$E(t) = \int u^2(y,t)\,dy$$

Error function

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-p^2}\,dp$$

Eigenvalues and eigenfunctions for Robin boundary problem $X'(0) = a_0 X(0)$, $X'(l) = -a_l X(l)$.

$$X_n(x) = \cos(\sqrt{\lambda_n}x) + \frac{a_0}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_n}x) \qquad \text{with } \lambda_n \text{ a root of} \qquad (\lambda_n - a_0 a_l)\tan(\sqrt{\lambda_n}l) = (a_0 + a_l)\sqrt{\lambda_n}$$

$$X_n(x) = \cosh(\sqrt{\lambda_n}x) + \frac{a_0}{\sqrt{\lambda_n}}\sinh(\sqrt{\lambda_n}x) \qquad \text{with } \lambda_n \text{ a root of} \qquad (\lambda_n + a_0 a_l)\tanh(\sqrt{\lambda_n}l) = -(a_0 + a_l)\sqrt{\lambda_n}$$

Polar form of the Laplacian in $\mathbb{R}^2$ and $\mathbb{R}^3$, valid for $r > 0$

$$\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$\Delta_3 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$